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Nondegeneracy condition of order q in discrete time

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Abstract

Arai (2007b) has obtained a sufficient condition under which the q-optimal martingale measure exists in discrete time, where $1 < q < \infty$. This sufficient condition is called the nondegeneracy condition. In this paper, the results in Arai (2007b) are introduced and two examples are illustrated, which one satisfies the nondegeneracy condition, and another does not.

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JEL classification: G10, C60

1 Introduction

The q-optimal martingale measure is defined as a signed martingale measure whose density minimizes the $\mathcal{L}^q$-norm. There are many literatures concerning the q-optimal martingale measure. For example, Grandits and Krawczyk (1998), Grandits (1999), Grandits and Rheinländer (2002), Hobson (2004), and so on. Moreover, Arai (2007a) has introduced the p-optimal hedging, which has much something to do with the q-optimal martingale measure, where p is the conjugate index of q, namely, we have $\frac{1}{q}+\frac{1}{p}=1$.

We firstly show how to calculate the density of the q-optimal martingale measure for discrete-time models. We introduce a predictable process β satisfying the following backward induction:

$$E\left[\Delta X_T\varphi_{p-1}(1-\beta_T\Delta X_T)|\mathcal{F}_{T-1}\right]=0,$$

and

$$E\left[\Delta X_k\varphi_{p-1}\left(\prod_{t=k}^{T}(1-\beta_t\Delta X_t)\right)|\mathcal{F}_{k-1}\right]=0,\ \text{for } t=1,\dots,T-1,$$

where, $T \in \mathbf{N}$ is the market maturity, X is the asset price process, and $\varphi_\alpha(x) := \mathrm{sgn}(x)|x|^\alpha$ for $\alpha > 0$ and $x \in \mathbf{R}$. We call β the adjustment process for X. The q-optimal martingale measure $Q^{(q)}$ then is given by

$$\frac{dQ^{(q)}}{dP} = \frac{\varphi_{p-1}\left(\prod_{t=1}^{T}(1-\beta_t\Delta X_t)\right)}{E[\varphi_{p-1}\left(\prod_{t=1}^{T}(1-\beta_t\Delta X_t)\right)]}.$$

Actually, Schweizer (1995, 1996) introduced this fact for the $\mathcal{L}^2$-case. On the other hand, Grandits(1999) studied this induction for the bounded asset price process case for $1 < q < \infty$. Moreover, Arai (2007b) dealt with this induction for the general $\mathcal{L}^q$-case. In the unbounded asset price process case, we have to pay attention to a sufficient condition under which there exists the q-optimal martingale measure. Note that the existence of the q-optimal martingale measure is ensured by the existence of a signed martingale measure whose density is in $\mathcal{L}^q$-space. He introduced the nondegeneracy condition and revealed the relationship with the closedness of the space of portfolio values and with the no-arbitrage condition. In this paper, we introduce the results of Arai (2007b) and two examples related to the nondegeneracy condition.

2 Preliminaries

Throughout this paper, we deal with a discrete time incomplete financial market with maturity $T \in \mathbf{N}$. Suppose that one riskless asset and only one risky asset are tradable. The price of the riskless asset is assumed to be given by 1 at all times. On the other hand, the fluctuation of the risky asset is expressed by a one-dimensional discrete time stochastic process X. Let $(\Omega, \mathcal{F}, P, \mathbf{F} = \{\mathcal{F}_t\}_{t=0,1,\dots,T})$ be a completed probability space. Suppose that X is $\mathbf{F}$-adapted. We denote $\Delta X_t = X_t - X_{t-1}$ and $G_t(\vartheta) = \sum_{i=1}^{t}\vartheta_i\Delta X_i$ for any predictable process ϑ. Next, we define some notations:

Definition 1 (1) We denote by Θ^p the set of predictable processes ϑ such that $\vartheta_k\Delta X_k \in \mathcal{L}^p(P)$ for any $k = 1,\dots,T$.
(2) We define $G_T^p = \{G_T(\vartheta)|\vartheta \in \Theta^p\}$.
(3) A signed measure Q is a signed martingale measure, if $Q \ll P$, $E\left[\frac{dQ}{dP}\right] = 1$, and $E\left[\frac{dQ}{dP}G_T(\vartheta)\right] = 0$ for all $\vartheta \in \Theta^p$.
(4) The set of all signed martingale measures is denoted by $\mathcal{M}^s$. In addition, $\mathcal{M}_q^s$ denotes the set of all signed martingale measures whose density with respect to P is in $\mathcal{L}^p(P)$.
(5) The q-optimal martingale measure $Q^{(q)}$ is defined as

$$Q^{(q)} = \arg \inf_{Q \in \mathcal{M}_q^s}\left\|\frac{dQ}{dP}\right\|_q.$$

We define two processes M and A as $A_k = E[X_k|\mathcal{F}_{k-1}]$ with $A_0 = 0$ and $M_k = X_k - X_0 - A_k$ for any $k = 0, \dots, T$. Note that M is a martingale with $M_0 = 0$ and A is predictable. Moreover, X is represented as $X = X_0 + M + A$. We now assume that there exists a predictable process λ satisfying $\Delta A_k = \lambda_k E[(\Delta M_k)^2|\mathcal{F}_{k-1}]$. In general, this condition is called the structure condition (SC). We finally define a norm on Θ^p as follows:

$$\|\vartheta\|_{\Theta^p} := \sum_{k=1}^{T} \|\vartheta_k \Delta X_k\|_p.$$

3 Closedness and no-arbitrage

Note that all results in this section are obtained in Arai (2007b). Hence, we omit all proofs.

We introduce the nondegeneracy condition (ND): there exists a constant $C > 0$ such that

$$|\Delta A_k|^q \leq C \frac{E^q[|\Delta M_k|^2|\mathcal{F}_{k-1}]}{E[|\Delta M_k|^q|\mathcal{F}_{k-1}]}.$$

We have the following three propositions:

Proposition 2 *Under the condition (ND), there exists a constant $C > 0$ such that $\|\vartheta\|_{\Theta^p} \leq C\|G_T(\vartheta)\|_p$, for any $\vartheta \in \Theta^p$.*

Proposition 3 *The operator $G_T(\cdot) : \Theta^p \to \mathcal{L}^p(P)$ is closed.*

Proposition 4 *Let V and W be two Banach spaces. Suppose that a closed operator $U : V \to W$ satisfies, for some constant $C > 0$,*

$$\|v\|_V \leq C\|Uv\|_W, \quad v \in D(U),$$

where $D(U)$ is the domain of U. Then, the region of U is a closed subspace.

We conclude the following theorem from the above Propositions 2, 3 and 4:

Theorem 5 *Under the condition (ND), G_T^p is $\mathcal{L}^p(P)$-closed.*

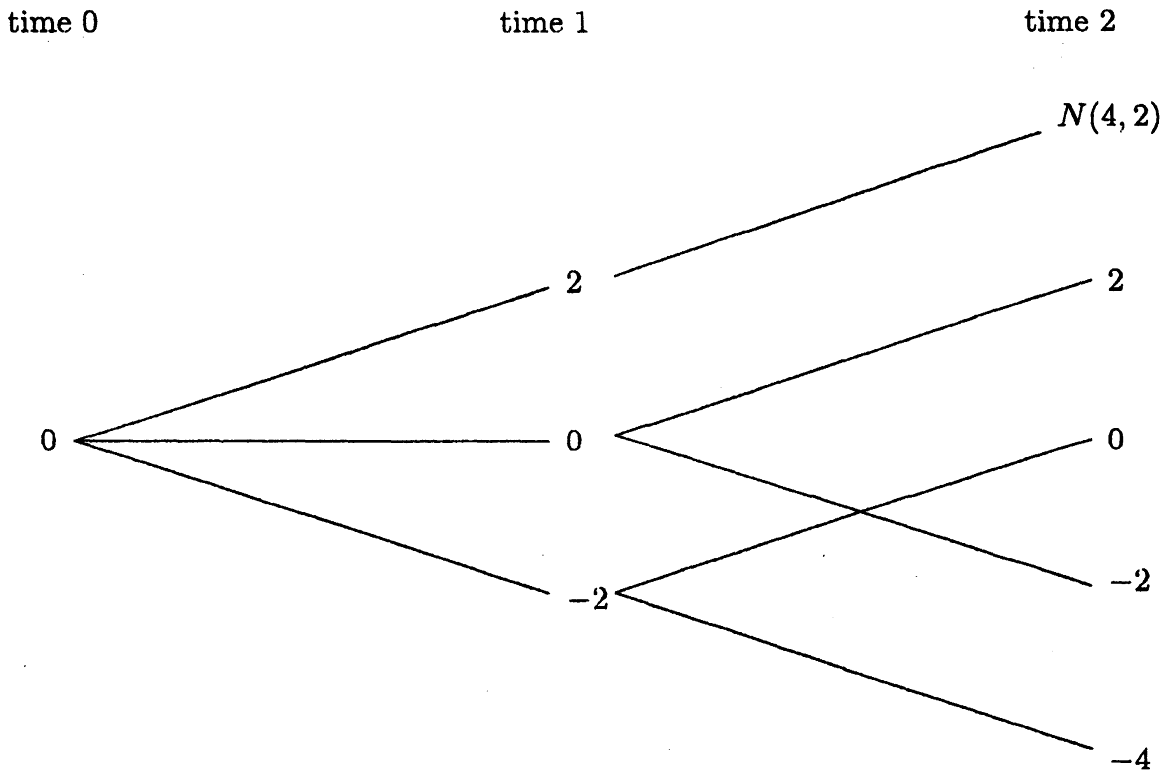
Finally, we introduce the relationship between the condition (ND) and the fundamental theorem of mathematical finance.

Theorem 6 *Under the condition (ND), $1 \notin G_T^p$ if and only if $\mathcal{M}_q^s \neq \emptyset$.*

4 Examples

We introduce two examples. The first example is one of typical unbounded models. We confirm that it satisfies the condition (ND). On the other hand, the second does not have the closedness of G_T^p. We make sure that it does not satisfy the condition (ND).

Example 7 We consider a 2-period model whose risky asset is not bounded. The asset price process X is given by the following figure.



We assume that $P(X_1 = 2) = P(X_1 = 0) = P(X_1 = -2)$ and $P(X_2 = 2|X_1 = 0) = P(X_2 = 0|X_1 = -2) = 1/2$. Note that the conditional distribution of ΔX_2 under $X_1 = 2$ follows $N(2, 2)$. In this model, $A_1 = 0$, and $A_2 = 0$ when $X_1 \neq 2$. When $X_1 = 2$, $A_2 = 2$. On the other hand, $M_1 = X_1$, and $M_2 = X_2$ when $X_1 \neq 2$. Moreover, if $X_1 = 2$, then ΔM_2 follows $N(0, 2)$. Thus, $|\Delta A_2|^q \leq 2^q$, and $E^q[|\Delta M_2|^2|X_1 = 2] = 2^q$. For $C \geq E[|\Delta M_2|^q|X_1 = 2] > 0$, the condition (ND) is satisfied.

Example 8 Consider a 3-period model, namely, set $T = 3$. Let U_1 and U_2 be independent two random variables which distribute uniformly on $[-1, 1]$. Moreover, let V be a $\{-1, 1\}$-valued random variable satisfying the following conditional probabilities:

$$P(V = 1|U_1) = |U_1|^p, \quad P(V = -1|U_1) = 1 - |U_1|^p.$$

Next, we define a filtration $\{\mathcal{F}_t\}_{t=0,1,2,3}$ as $\mathcal{F}_0$ =trivial, $\mathcal{F}_1 = \sigma(U_1)$, $\mathcal{F}_2 = \sigma(U_1) \vee \sigma(U_2)$, and $\mathcal{F}_3 = \sigma(U_1) \vee \sigma(U_2) \vee \sigma(V)$. The asset price process X is defined as $X_0 = 0$, $\Delta X_1 = U_1$, $\Delta X_2 = U_2$ and

$$\Delta X_3 = \begin{cases} |U_1| & \text{if } V = 1, \\ -1 & \text{if } V = -1. \end{cases}$$

Now, we define a predictable process ξ as

$$\xi_1 = 0, \quad \xi_2 = |U_1|^{-1}, \quad \xi_3 = U_2/|U_1|.$$

Then, $G_3(\xi) \in \mathcal{L}^p(P)$ and $\xi \notin \Theta^p$. In addition, when we denote $\xi^n := \xi 1_{\{|U_1| \geq 1/n\}}$, we have that $\xi^n \in \Theta^p$ and $G_3(\xi^n) \to G_3(\xi)$ in $\mathcal{L}^p(P)$. In summary, Proposition 2 does not hold.

Actually, our model does not satisfy the condition (ND). Let us confirm this fact. We firstly remark that

$$|\Delta A_3|^q = ||U_1|^{1+p} + |U_1|^p - 1|^q$$

and the right hand side of (ND) at $k = 3$ is given by

$$(1 + |U_1|)^q (1 - |U_1|)^{q-1} |U_1|^q ((1 - |U_1|^p)^{q-1} + |U_1|^q)^{-1}.$$

When $|U_1|$ tends to 0 or 1, the left hand side of (ND) converges to 1 and the right hand side 0. Thus, there is no constant $C > 0$ satisfying the condition (ND).

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